Mixed-state additivity properties of magic monotones based on quantum relative entropies for single-qubit states and beyond

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Joint work with Roberto Rubboli and Marco Tomamichel

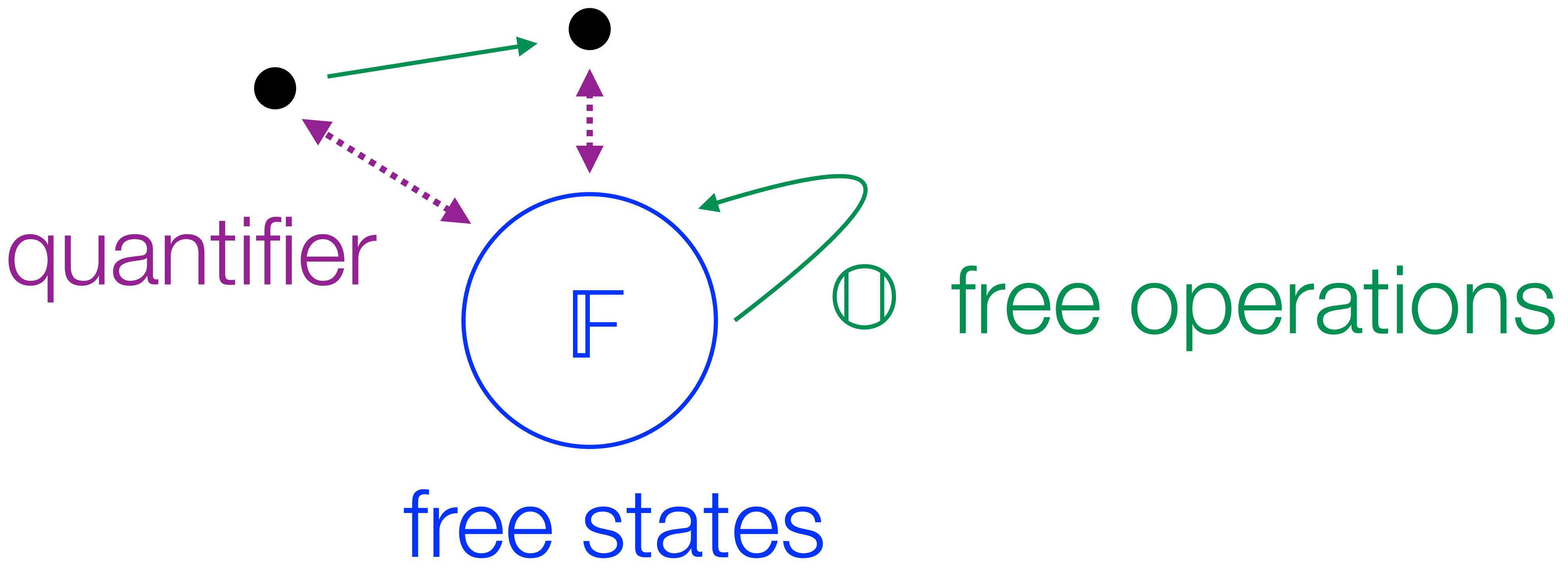
[Quantum 8, 1492 (2024)]

Introduction

Quantum resource theories and magic measures

Big goal: Quantitative understanding of quantum resources enabling quantum advantages underlying given physical and operational settings.

Quantum resource theories: Framework to deal with **quantification** and **manipulation** of quantum resources



[Chitambar, Gour, Rev. Mod. Phys. '19]

- Quantum entanglement

  [Horodecki et al. Rev. Mod. Phys. '09]

- Quantum thermodynamics

  [Horodecki, Oppenheim, Nat. Commun. '13]

Here, $\mathbb{F}$: stabilizer states, $\mathbb{O}$: stabilizer protocols

Good resource measures…?

Necessary property for resource measure: monotonicity

$$M(\rho) \geq M(\Lambda(\rho)) \qquad \forall \Lambda \in \mathbb{O}$$

There are infinitely many monotones. What are "good" ones?

- Computable (e.g., stabilizer Renyi entropy for pure states)
- Operational interpretation (e.g., robustness)
- Useful in restricting state transformation $\rho \underset{\mathbb{O}}{\rightarrow} \tau$

Additive measure $M(\rho_1 \otimes \rho_2) = M(\rho_1) + M(\rho_2)$ **is useful**

Great interest in finding additive measure of entanglement

[Christandl, Winter, J. Math. Phys. '04]

Implications to state transformation

Asymptotic transformation $\rho^{\otimes n} \underset{\mathbb{O}}{\to} \tau^{\otimes m}$ $\quad M(\rho^{\otimes n}) \geq M(\tau^{\otimes m})$

$$M(\rho_1 \otimes \rho_2) = M(\rho_1) + M(\rho_2)$$

Minimum n required (converse of asymptotic rate) $\quad nM(\rho) \geq M(\rho^{\otimes n}) \geq M(\tau^{\otimes m}) = mM(\tau)$

subadditivity (easier to show) — for $nM(\rho) \geq M(\rho^{\otimes n})$

additivity — for $M(\tau^{\otimes m}) = mM(\tau)$

$$\frac{n}{m} \geq \frac{M(\rho)}{M(\tau)}$$

Catalytic transformation $\rho \otimes \eta \underset{\mathbb{O}}{\to} \tau \otimes \eta$ $\quad M(\rho \otimes \eta) \geq M(\tau \otimes \eta)$

If M is additive, $\quad M(\rho) + M(\eta) \geq M(\rho \otimes \eta) \geq M(\tau \otimes \eta) = M(\tau) + M(\eta)$

additivity — for $M(\tau \otimes \eta) = M(\tau) + M(\eta)$

$\Rightarrow$

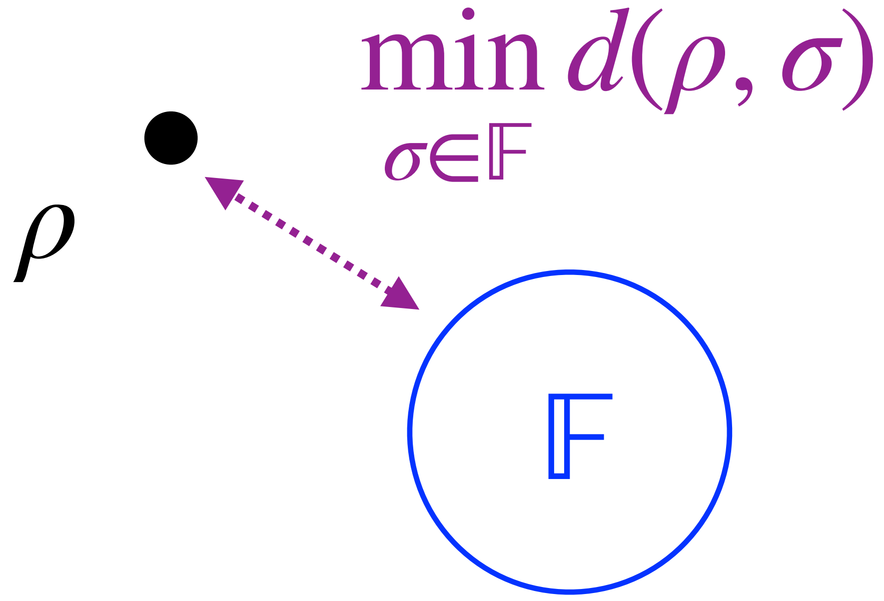
$$M(\rho) \geq M(\tau)$$

Can magic measure be additive?

Some measures are known to be additive [Seddon et al. PRX Quantum '21]

We add a new family of additive magic measures and get new bounds for state transformation

Stabilizer fidelity



$\min_{\sigma\in\mathbb{F}} d(\rho,\sigma)$ is a resource monotone if d satisfies data-processing

⟶ Take $d(\rho,\sigma) = \log F(\rho,\sigma)^{-1}$ $\quad F(\rho,\sigma) = \left(\mathrm{Tr}\sqrt{\sigma^{1/2}\rho\sigma^{1/2}}\right)^2$

$$\mathscr{D}_{\mathrm{fid}}(\rho) := \log \mathscr{F}(\rho)^{-1}$$

$\mathscr{F}(\rho) := \max_{\sigma\in\mathrm{STAB}} F(\rho,\sigma)$ **stabilizer fidelity** [Bravyi et al., Quantum '19]

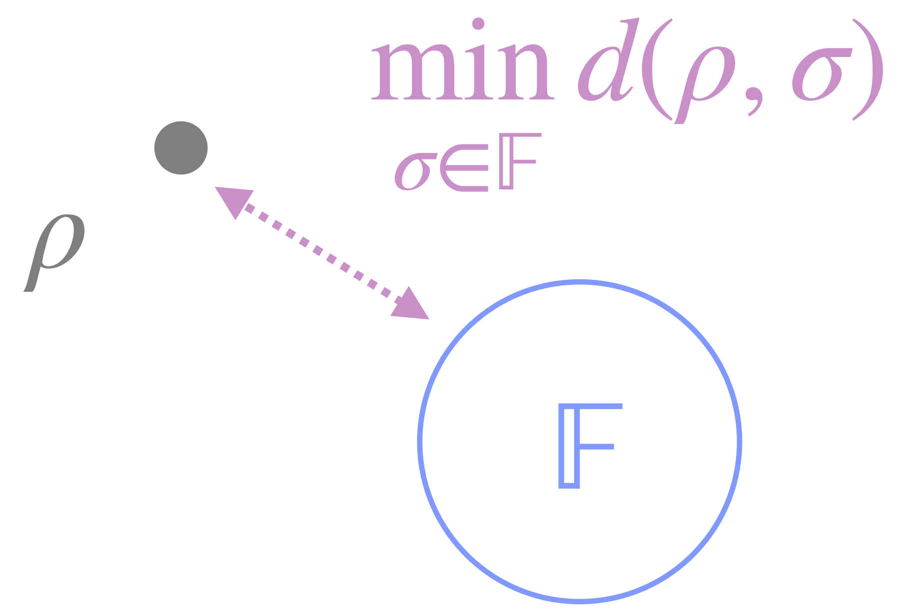
Subadditivity $\mathscr{D}_{\mathrm{fid}}(\rho_1\otimes\rho_2) \leq \mathscr{D}_{\mathrm{fid}}(\rho_1) + \mathscr{D}_{\mathrm{fid}}(\rho_2)$ is easy to see

Letting $\sigma_i \in \mathrm{STAB}$ satisfy $\mathscr{D}_{\mathrm{fid}}(\rho_i) = \log F(\rho_i,\sigma_i)^{-1}$,

$$\mathscr{D}_{\mathrm{fid}}(\rho_1\otimes\rho_2) \leq \log F(\rho_1\otimes\rho_2, \sigma_1\otimes\sigma_2)^{-1} = \log F(\rho_1,\sigma_1)^{-1} + \log F(\rho_2,\sigma_2)^{-1} = \mathscr{D}_{\mathrm{fid}}(\rho_1) + \mathscr{D}_{\mathrm{fid}}(\rho_2)$$

Superadditivity $\mathscr{D}_{\mathrm{fid}}(\rho_1\otimes\rho_2) \geq \mathscr{D}_{\mathrm{fid}}(\rho_1) + \mathscr{D}_{\mathrm{fid}}(\rho_2)$ would imply additivity

Stabilizer fidelity



$\min_{\sigma \in \mathbb{F}} d(\rho, \sigma)$ is a resource monotone if d satisfies data-processing

$\longrightarrow$ Take $d(\rho, \sigma) = \log F(\rho, \sigma)^{-1}$ $\quad F(\rho, \sigma) = \left(\mathrm{Tr} \sqrt{\sigma^{1/2} \rho \sigma^{1/2}} \right)^2$

$\mathscr{D}_{\mathrm{fid}}(\rho) := \log \mathscr{F}(\rho)^{-1}$ $\quad \mathscr{F}(\rho) := \max_{\sigma \in \mathrm{STAB}} F(\rho, \sigma)$ **stabilizer fidelity** [Bravyi et al., Quantum '19]

Superadditivity $D_{\mathrm{fid}}(\rho_1 \otimes \rho_2) \geq \mathscr{D}_{\mathrm{fid}}(\rho_1) + \mathscr{D}_{\mathrm{fid}}(\rho_2)$ is non-trivial.

$\sigma_{12} \in \mathrm{STAB}$ such that $\mathscr{D}_{\mathrm{fid}}(\rho_1 \otimes \rho_2) = \log F(\rho_1 \otimes \rho_2, \sigma_{12})^{-1}$ may not be product $\sigma_{12} = \sigma_1 \otimes \sigma_2$

For **pure states** $\psi_{1,2}$ up to three qubits, $\mathscr{D}_{\mathrm{fid}}(\psi_1 \otimes \psi_2) \geq \mathscr{D}_{\mathrm{fid}}(\psi_1) + \mathscr{D}_{\mathrm{fid}}(\psi_2)$ **additivity!**

What about mixed states? [Bravyi et al., Quantum '19]

Magic measures with quantum relative entropies

Magic measures with $\alpha - z$ Renyi divergences

$$D_{\alpha,z}(\rho\|\sigma) = \frac{1}{\alpha-1}\log \mathrm{Tr}\left[\sigma^{\frac{1-\alpha}{2z}}\rho^{\frac{\alpha}{z}}\sigma^{\frac{1-\alpha}{2z}}\right]^{z}$$

[Audenaert, Datta, JMP '15]

$z = 1$ Petz divergence

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log \mathrm{Tr}(\rho^\alpha\sigma^{1-\alpha})$$

$z = \alpha$ Sandwiched divergence

$$\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log \mathrm{Tr}\left[\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right]^{\alpha}$$

Limiting cases

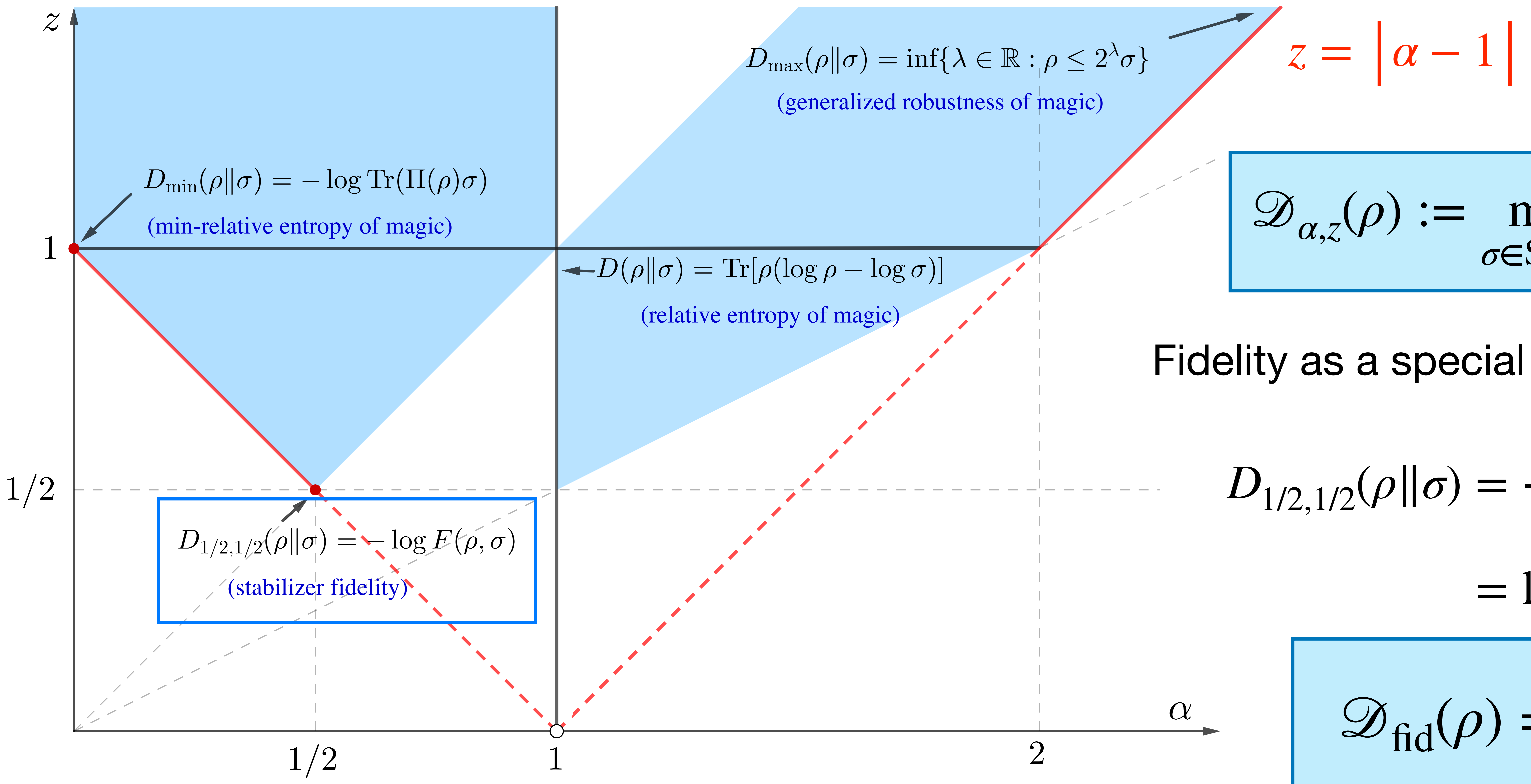
$$D_{\min}(\rho\|\sigma) = -\log \mathrm{Tr}[\Pi(\rho)\sigma] = \lim_{\alpha\to 0} D_{\alpha,1-\alpha}(\rho\|\sigma)$$

$$D(\rho\|\sigma) = \mathrm{Tr}\rho(\log\rho - \log\sigma) = \lim_{\alpha\to 1} D_{\alpha,\alpha}(\rho\|\sigma)$$

$$D_{\max}(\rho\|\sigma) = \log\|\sigma^{-1/2}\rho\sigma^{-1/2}\|_\infty = \lim_{\alpha\to\infty} D_{\alpha,\alpha-1}(\rho\|\sigma)$$

$$D_{\alpha,z}(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \mathrm{Tr}\left[\sigma^{\frac{1-\alpha}{2z}} \rho^{\frac{\alpha}{z}} \sigma^{\frac{1-\alpha}{2z}}\right]^z$$

Blue: "data-processing inequality region"



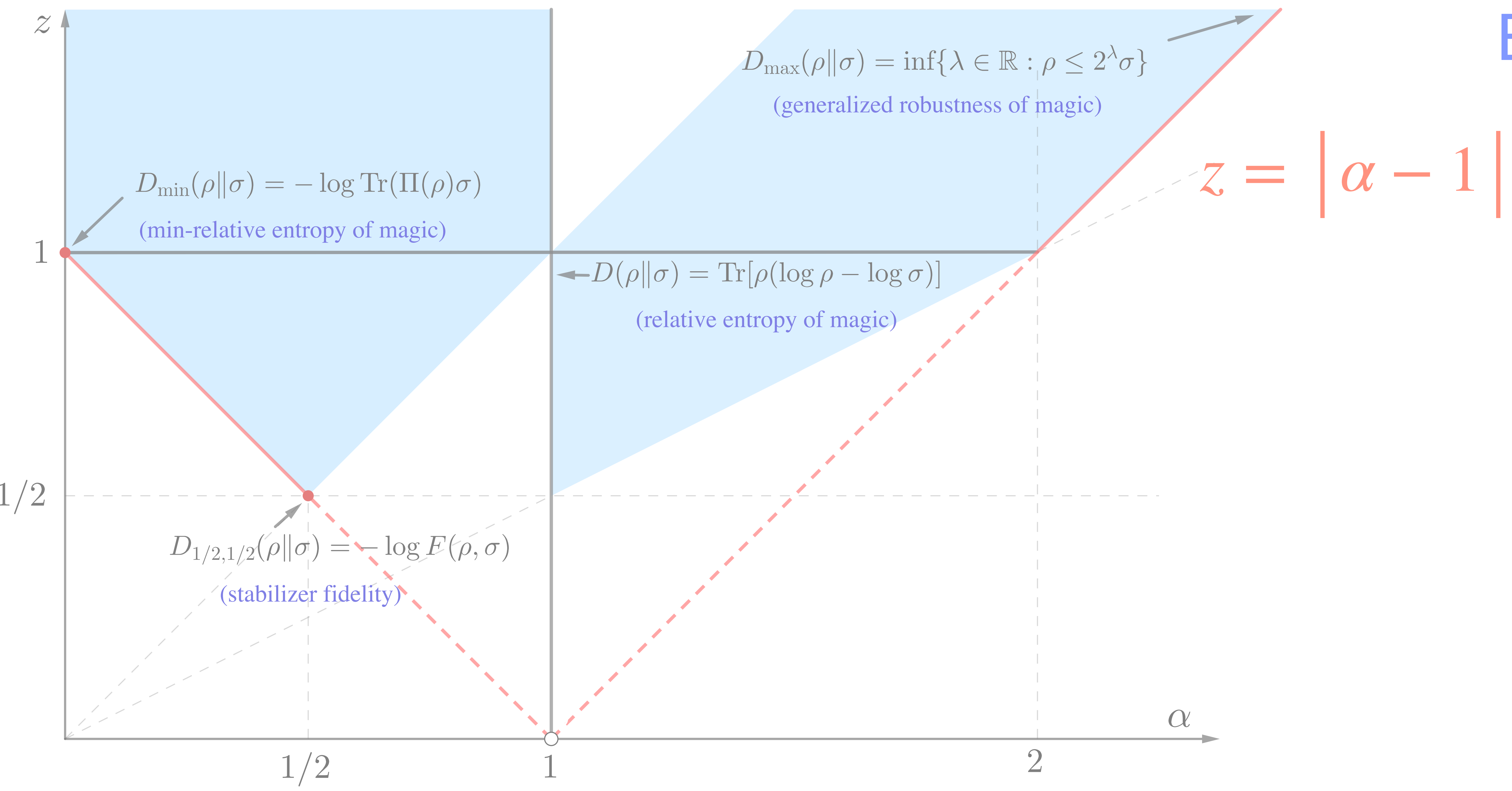
$$\mathscr{D}_{\alpha,z}(\rho) := \min_{\sigma \in \mathrm{STAB}} D_{\alpha,z}(\rho\|\sigma)$$

Fidelity as a special case on $z = |\alpha - 1|$

$$D_{1/2,1/2}(\rho\|\sigma) = -2\log \mathrm{Tr}\sqrt{\sigma^{1/2}\rho\sigma^{1/2}}$$
$$= \log F(\rho,\sigma)^{-1}$$

$$\mathscr{D}_{\mathrm{fid}}(\rho) = \mathscr{D}_{1/2,1/2}(\rho)$$

Additivity of single-qubit states



Blue: "data-processing inequality region"

$$D_{\alpha,z}(\rho\|\sigma) = \frac{1}{\alpha-1}\log \mathrm{Tr}\left[\sigma^{\frac{1-\alpha}{2z}}\rho^{\frac{\alpha}{z}}\sigma^{\frac{1-\alpha}{2z}}\right]^z$$

$$\mathscr{D}_{\alpha,z}(\rho) := \min_{\sigma\in\mathrm{STAB}} D_{\alpha,z}(\rho\|\sigma)$$

$$\mathscr{D}_{\mathrm{fid}}(\rho) = \mathscr{D}_{1/2,1/2}(\rho)$$

For any single-qubit states $\{\rho_i\}_i$, $\mathscr{D}_{\alpha,z}(\otimes_i\rho_i) = \sum_i \mathscr{D}_{\alpha,z}(\rho_i)$ for $z = |\alpha - 1|$

Extending pure-state additivity of stabilizer fidelity

Proof idea

$$D_{\alpha,z}(\rho\|\sigma) = \frac{1}{\alpha-1}\log \mathrm{Tr}\left[\sigma^{\frac{1-\alpha}{2z}}\rho^{\frac{\alpha}{z}}\sigma^{\frac{1-\alpha}{2z}}\right]^z \qquad \mathscr{D}_{\alpha,z}(\rho) := \min_{\sigma\in\mathrm{STAB}} D_{\alpha,z}(\rho\|\sigma)$$

Core part of the issue

$\mathscr{D}_{\alpha,z}(\rho_1\otimes\rho_2) = D_{\alpha,z}(\rho_1\otimes\rho_2\|\sigma_{12})$ Does the optimizer have the form $\sigma_{12} = \sigma_1\otimes\sigma_2$?

Condition for state σ to be an optimizer of $\min_{\sigma\in\mathrm{STAB}} D_{\alpha,z}(\rho\|\sigma)$

σ is an optimizer if and only if $\mathrm{Tr}(\tilde\sigma\,\Xi_{\alpha,z}(\rho)) \le Q_{\alpha,z}(\rho\|\sigma) \quad \forall\tilde\sigma\in\mathrm{STAB}$

$Q_{\alpha,z}(\rho\|\sigma) := \exp\left[(\alpha-1)D_{\alpha,z}(\rho\|\sigma)\right]$ $\Xi(\rho,\sigma)$: operator depending on ρ,σ

cf. [Rubboli, Tomamichel, Commun. Math. Phys. ’22]

This is linear in $\tilde\sigma\in\mathrm{STAB}$. Much simpler than original nonlinear minimization.

Proof idea

$$D_{\alpha,z}(\rho\|\sigma) = \frac{1}{\alpha-1}\log \mathrm{Tr}\left[\sigma^{\frac{1-\alpha}{2z}}\rho^{\frac{\alpha}{z}}\sigma^{\frac{1-\alpha}{2z}}\right]^z \qquad \mathscr{D}_{\alpha,z}(\rho) := \min_{\sigma\in\mathrm{STAB}} D_{\alpha,z}(\rho\|\sigma)$$

Core part of the issue

$$\mathscr{D}_{\alpha,z}(\rho_1\otimes\rho_2) = D_{\alpha,z}(\rho_1\otimes\rho_2\|\sigma_{12})$$

Does the optimizer have the form $\sigma_{12} = \sigma_1\otimes\sigma_2$?

σ is an optimizer if and only if $\mathrm{Tr}(\tilde\sigma\,\Xi_{\alpha,z}(\rho,\sigma)) \le Q_{\alpha,z}(\rho\|\sigma) \quad \forall\tilde\sigma\in\mathrm{STAB}$

$$Q_{\alpha,z}(\rho\|\sigma) := \exp\left[(\alpha-1)D_{\alpha,z}(\rho\|\sigma)\right]$$

Take optimizers σ_1 and σ_2 for $\mathscr{D}_{\alpha,z}(\rho_1)$ and $\mathscr{D}_{\alpha,z}(\rho_2)$, and check if $\sigma_1\otimes\sigma_2$ is an optimizer

$$\mathrm{Tr}(\tilde\sigma\,\Xi_{\alpha,z}(\rho_1\otimes\rho_2,\sigma_1\otimes\sigma_2)) \le Q_{\alpha,z}(\rho_1\otimes\rho_2\|\sigma_1\otimes\sigma_2) = Q_{\alpha,z}(\rho_1\|\sigma_1)Q_{\alpha,z}(\rho_2\|\sigma_2) \quad \forall\tilde\sigma\in\mathrm{STAB}$$

multiplicativity of $Q_{\alpha,z}$ (easy to show)

$\Xi_{\alpha,z}(\rho_1\otimes\rho_2,\sigma_1\otimes\sigma_2) = \Xi_{\alpha,z}(\rho_1,\sigma_1)\Xi_{\alpha,z}(\rho_2,\sigma_2)$ for $z = |\alpha-1|$

Proof idea

$$D_{\alpha,z}(\rho\|\sigma) = \frac{1}{\alpha-1}\log \mathrm{Tr}\left[\sigma^{\frac{1-\alpha}{2z}}\rho^{\frac{\alpha}{z}}\sigma^{\frac{1-\alpha}{2z}}\right]^z \quad \mathscr{D}_{\alpha,z}(\rho) := \min_{\sigma\in\mathrm{STAB}} D_{\alpha,z}(\rho\|\sigma)$$

$$Q_{\alpha,z}(\rho\|\sigma) := \exp\left[(\alpha-1)D_{\alpha,z}(\rho\|\sigma)\right]$$

Using $\Xi_{\alpha,z}(\rho_1\otimes\rho_2,\sigma_1\otimes\sigma_2) = \Xi_{\alpha,z}(\rho_1,\sigma_1)\Xi_{\alpha,z}(\rho_2,\sigma_2)$ for $z = \left|\alpha-1\right|$

$$\mathrm{Tr}(\tilde\sigma\,\Xi_{\alpha,z}(\rho_1\otimes\rho_2,\sigma_1\otimes\sigma_2)) \le Q_{\alpha,z}(\rho_1\otimes\rho_2\|\sigma_1\otimes\sigma_2) = Q_{\alpha,z}(\rho_1\|\sigma_1)Q_{\alpha,z}(\rho_2\|\sigma_2) \quad \forall\tilde\sigma\in\mathrm{STAB}$$

reduces to

$$\max_{\sigma\in\mathrm{STAB}} \mathrm{Tr}(\sigma_{12}\,\tilde\rho_1\otimes\tilde\rho_2) = \max_{\sigma_1\in\mathrm{STAB}} \mathrm{Tr}(\sigma_1\,\tilde\rho_1)\max_{\sigma_2\in\mathrm{STAB}} \mathrm{Tr}(\sigma_2\,\tilde\rho_2),\ \forall\tilde\rho_1,\tilde\rho_2$$

which can be shown for single-qubit states $\tilde\rho_1$ and $\tilde\rho_2$.

More additivity for standard magic states

We can extend additivity to general α and z for some class of magic states.

$\Delta_p(\rho) := (1-p)\rho + p\mathbb{I}/d$: depolarizing channel

Let $\{\rho_i\}_i$ be states on at most three-qubit systems such that $\rho_i = \Delta_{p_i}(\psi_i)$ for some pure state ψ_i and some $p_i \geq 0$. If the optimizer σ_i such that $\mathscr{D}_{\alpha,z}(\rho_i) = D_{\alpha,z}(\rho_i \| \sigma_i)$ is a depolarized state from ρ_i such that $\Delta_{s_i}(\sigma_i)$ for some $s_i \geq 0$, then it holds that

$$\mathscr{D}_{\alpha,z}(\otimes_i \rho_i) = \sum_i \mathscr{D}_{\alpha,z}(\rho_i)$$

for any α and z in the “data-processing region”.

This includes $T = \frac{1}{2}\left(\mathbb{I} + \frac{X+Y}{\sqrt{2}}\right)$, $F = \frac{1}{2}\left(\mathbb{I} + \frac{X+Y+Z}{\sqrt{3}}\right)$, $\left|\mathrm{Toffoli}\right\rangle = \mathrm{Toffoli}\left|++0\right\rangle$

Applications

Asymptotic magic state transformation rate

$$R(\rho \to \tau) := \sup \left\{ r \,\middle|\, \rho^{\otimes n} \xrightarrow[\text{STAB}]{\epsilon_n} \tau^{\otimes rn},\ \lim_{n\to\infty} \epsilon_n = 0 \right\}$$

$$R(\rho \to \tau) \leq \frac{\mathscr{D}^\infty(\rho)}{\mathscr{D}^\infty(\tau)}$$

$$\mathscr{D}^\infty(\rho) := \lim_{n\to\infty} \frac{\mathscr{D}(\rho^{\otimes n})}{n} \qquad \mathscr{D}(\rho) := \min_{\sigma\in\text{STAB}} D(\rho\|\sigma)$$

[Horodecki, Oppenheim, Int J. Mod. Phys. B ’13]

Computable upper bound?

$\mathscr{D}^\infty(\rho) \leq \mathscr{D}(\rho)$ (easy)

additivity!

$$\mathscr{D}^\infty(\tau) \geq \lim_{n\to\infty} \frac{\mathscr{D}_{1/2,1/2}(\tau^{\otimes n})}{n} = \mathscr{D}_{1/2,1/2}(\tau) = \log \mathscr{F}(\tau)^{-1}$$

$$R(\rho \to \tau) \leq \frac{\mathscr{D}(\rho)}{\log \mathscr{F}(\tau)^{-1}}$$

τ: single-qubit state

Improves the previous best single-copy bound.

[Seddon et al., PRX Quantum ’21]

Deterministic and probabilistic magic state manipulation

$$\rho^{\otimes n} \xrightarrow[\text{STAB}]{} \tau \text{ with probability } p$$

$$D_{\alpha,z}(\rho\|\sigma) = \frac{1}{\alpha-1}\log \mathrm{Tr}\left[\sigma^{\frac{1-\alpha}{2z}}\rho^{\frac{\alpha}{z}}\sigma^{\frac{1-\alpha}{2z}}\right]^z$$

$$\mathscr{D}_{\alpha,z}(\rho) := \min_{\sigma\in\text{STAB}} D_{\alpha,z}(\rho\|\sigma)$$

$$Q_{\alpha,z}(\rho\|\sigma) := \exp\left[(\alpha-1)D_{\alpha,z}(\rho\|\sigma)\right]$$

$$n\mathscr{D}_{\alpha,z}(\rho) \geq \frac{\alpha}{\alpha-1}\log\left[p\,\mathscr{Q}_{\alpha,z}^{\frac{1}{\alpha}}(\tau) + (1-p)\right]$$

$$\mathscr{Q}_{\alpha,z}(\rho) := \exp\left[(\alpha-1)\mathscr{D}_{\alpha,z}(\rho)\right]$$

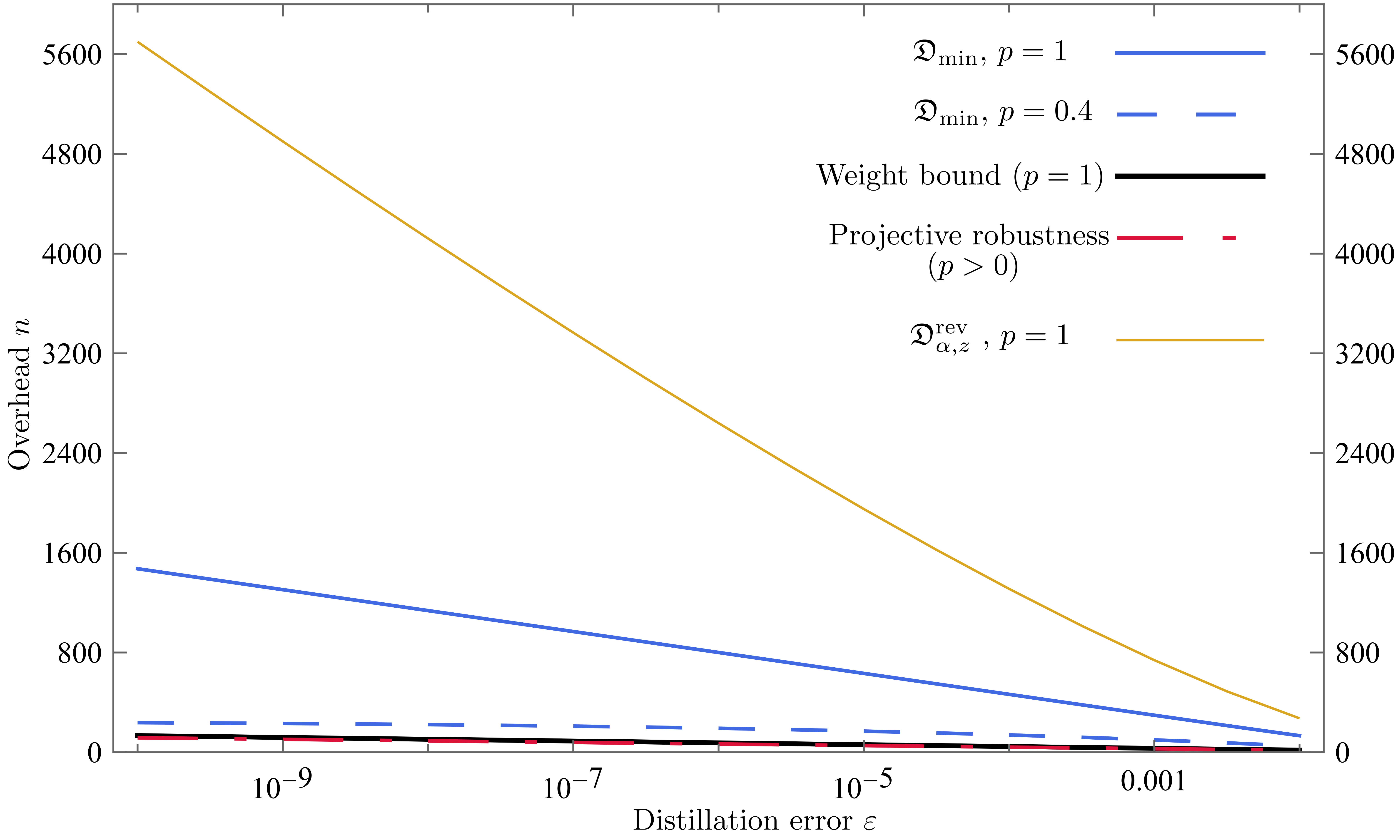
Implication of additivity for $\mathscr{D}_{\alpha,z}$: $\mathscr{Q}_{\alpha,z}(\tau^{\otimes m}) = \left[\mathscr{Q}_{\alpha,z}(\tau)\right]^m$ Right-hand side computable

Another application of our measure : new bound for magic state distillation

$$T_\delta := (1-\delta)\left|T\rangle\langle T\right| + \delta\frac{\mathbb{I}}{2} \qquad T_\delta^{\otimes n} \xrightarrow[\text{STAB}]{\varepsilon} \left|T\rangle\langle T\right| \text{ with probability } p$$

Our result provides a lower bound for the overhead n

Deterministic and probabilistic magic state manipulation



$$T_\delta^{\otimes n} \xrightarrow[\rm STAB]{\varepsilon} |T\rangle\langle T|$$

with probability p

$$T_\delta := (1-\delta)|T\rangle\langle T| + \delta\frac{\mathbb{I}}{2}$$

Here we set $\delta = \frac{3}{4}$

Another measure we introduce

$$\mathscr{D}^{\rm rev}_{\alpha,z}(\rho) = \min_{\sigma\in{\rm STAB}} D_{\alpha,z}(\sigma\|\rho)$$

Bounds with $\mathscr{D}_{\min}$: $\mathscr{D}_{\alpha,1-\alpha}$ with $\alpha \to 0$ and $\mathscr{D}^{\rm rev}_{\alpha,z}$ outperform previous ones

[Regula, Takagi, Nat. Commun. '21] [Fang, Liu, PRXQ '22] [Regula, PRL '22]

Conclusions and outlook

- New family of additive magic measure including stabilizer fidelity and $D_{\max}$ measure

- Additivity shown by "linearization" technique characterizing minimizer of divergences

- Put new fundamental limitations on magic state transformation

- Extend the additivity to all states on at most three-qubit systems.

- Tensor product additive, superadditive faithful magic measure?

Thank you!